ISSN 0460-2242

LIETUVOS MECHANIKOS RINKINYS

LITHUANIAN JOURNAL OF COMPUTATIONAL MECHANICS

VOL. 33, 1994

VILNIUS "TECHNIKA" 1994

Conclusions

This paper presents a technique for the prediction of the local and effective behavior of piezoelectric composites with a periodic structure by using the equations of the linear Toupin's piezoelectricity theory. For the calculation of the local and effective coefficients, it is necessary to solve some periodic problems on a cell. For laminated composites, these problems are solved exactly, so the local and effective coefficients of piezoelectric laminates are obtained in an explicit form.

References

1. N.S.Bakhvalov and G.P.Panasenko, Homogenization of processes in periodic structures, Moscow, Nauka, (1984) (in Russian).
2. B.E.Pobedria, Mechanics of composite materials, Moscow University Press, Moscow, (1984) (in Russian).
3. R.A.Toupin, The elastic dielectrics, J.Rational Mech. and Analysis, 5(6) (1956) 849-915.
4. W.Voigt, Lehrbuch der Kristall-Physik, Teubner, Leipzig, (1910).
5. R.Karaliūnas, Homogenization of equations of thermopiezoelectric composites, Lietuvos Matematikos Rinkinys, 30(1) (1990) 56-70 (in Russian).
6. R.J.Karaliūnas, Effective thermopiezoelectric properties of laminated composites, Mekhanika Kompozitnykh Materialov, 5 (1990) 823-830 (in Russian).
7. L.P.Khoroshun, B.P.Maslov and P.V.Leshchenko, Prediction of effective properties of piezoactive composite materials, Kiev, Naukova Dumka, (1989) (in Russian).

Received 8 May 1992

DYNAMIC ANALYSIS OF ELASTIC STRUCTURES WITH NONLINEAR INTERACTION POINTS - MOTION LAWS WITH STEADY AND SLOWLY VARYING AMPLITUDES

Rimantas Barauskas
Kaunas University of Technology

The paper presents techniques for the analysis of resonant elastic mechanical structures with nonlinear interactions vibrating with steady or slow-varying amplitudes. A motion law is expressed as a superposition of some set of orthogonal periodic functions by employing the weighted residual techniques. The harmonic balance method is shown to be a special case of the approach. For obtaining the motion law analysis in terms of slow varying amplitudes the time-averaging techniques are employed. When integrating the averaged equations of motion numerically, harmonic components at each time station are obtained by means of the Fourier transformation. The text is supplied with a numerical example.

Introduction

The displacement approach of the finite element method used for both the continuous mechanical systems and those with lumped parameters leads to the uniform model equations of motion

$$M\ddot{U}+C\dot{U}+KU=W\left(U,\dot{U}\right)+R(t), \tag{1}$$

where M,C,K - the matrices of the elastic structure, $R(t)$ - the exciting force vector, $W\left(U,\dot{U}\right)$ - the nonlinear force vector.

The steady motion laws of elastic structures as well as the transient ones can be obtained by the direct integration of the equations of motion. If damping forces are present, transient motions taking place after an external force is applied to a structure cease after a certain time interval t^{*}. The motion law after the time point t^{*} can reasonably be regarded as steady. Unfortunately, for structures with high values of the mechanical Q-factor (high-quality resonant structures) such an approach is very inefficient and it can even lead to incorrect results because of a very large number of integration steps until the steady motion law is obtained. In some cases even the existence of periodic motion corresponding to a given excitation law remains unclear. The principal source of rounding-off errors is an essential difference between the magnitudes of conservative and dissipative terms in the equations of motion. The inertia force $M\ddot{U}$ and the elastic force KU predominate over the dissipative force $C\dot{U}$, $\left|M\ddot{U}\right|>>\left|C\dot{U}\right|$, $\left|KU\right|>>\left|C\dot{U}\right|$, but the dynamic equilibrium condition implies the relation $\left|M\ddot{U}+KU\right|\approx\left|C\dot{U}\right|$, resulting in poor accuracy of the computer arithmetics.

There are several approaches enabling to avoid or to reduce the difficulties mentioned above, e.g., [1,2,3]. As a rule, they can be applied directly to structures with only several degrees of freedom. Regarding large structures, some development is necessary. The analysis of steady motion laws in the time domain can be carried out by finding the zero values of some algebraic function obtained by integrating directly the equations of motion [4,5]. In the frequency domain, nonlinear harmonic balance equations are obtained, and the use of the averaging techniques enables to investigate transient motions in terms of slow-varying amplitudes [1,6,7]. In [7] a combined harmonic balance and direct integration approach has been applied to structures with local nonlinearities.

In this paper a formulation obtained by employing weighted residuals is presented, the harmonic balance method being a special case of the approach under consideration. Numerical examples present high-frequency impact vibration laws of a vibroconverter attached to a rigid plane by means of a constant loading force, and the results are compared with the motion laws obtained by employing direct time integration techniques.

The weighted residual approach for obtaining steadyvibration laws

Consider the matrix equation of motion of an elastic structure with nonlinear interaction as (1) with the periodic excitation $R(t)=R(t+T)$. The motion law $U(t)$ of the structure during the vibration period is expressed by superimposing a set of some T-periodic time functions as

$$U(t)=N(t)U_A, \tag{2}$$

where $N(t)$ - the matrix containing time functions, and U_A - a constant vector of the generalized nodal amplitudes. By substituting (2) into (1) and by weighting the residual during the period the nonlinear equation is expressed in terms of the generalized amplitudes U_A :

$$AU_A=\int_0^T N^{\mathrm{T}}W\left(NU_A,\dot{N}U_A\right)\mathrm{d}t+\int_0^T N^{\mathrm{T}}R(t)\,\mathrm{d}t\,, \tag{3}$$

where $A=\int_0^T\left(N^{\mathrm{T}}M\ddot{N}+N^{\mathrm{T}}C\dot{N}+N^{\mathrm{T}}KN\right)\mathrm{d}t$.

Nonlinear algebraic equation (3) can be derived by employing simple iteration as well as the Newton-Raphson iteration scheme, the latter being presented as

$$U_A^{i+1}=U_A^i+A_{\mathrm{T}}^{-1}\left(R_A-AU_A^i+R_N\right), \tag{4}$$

where

$$\begin{gathered}A_{\mathrm{T}}=A-\int_0^T N^{\mathrm{T}}\frac{\partial W}{\partial U}\bigg|_i N\,\mathrm{d}t-\int_0^T N^{\mathrm{T}}\frac{\partial W}{\partial \dot{U}}\bigg|_i \dot{N}\,\mathrm{d}t,\\ R_N=\int_0^T N^{\mathrm{T}}W\left(NU_A^i,\dot{N}U_A^i\right)\mathrm{d}t,\quad R_A=\int_0^T N^{\mathrm{T}}R(t)\mathrm{d}t,\end{gathered} \tag{5}$$

and the notation $|_i$ means "obtained by substituting the values NU_A^i, $\dot{N}U_A^i$".

Application to unilaterally constrained structures

Consider a linear structural equation of motion with unilateral constraints

$$\begin{cases}M\ddot{U}+C\dot{U}+KU=R(t),\\ PU\le d.\end{cases} \tag{6}$$

Substituting relation (2) into (6) and weighting the residual, we obtain

$$\begin{cases}AU_A=R_A\\ PNU_A\le d.\end{cases} \tag{7}$$

The constraints of set (7) are to be satisfied at each time point during the vibration period. By introducing the Lagrange multiplier vector $\lambda(t)\ge 0$, , the following set of equations is obtained:

$$\begin{cases}AU_A+\int_0^T N^{\mathrm{T}}P^{\mathrm{T}}\lambda\,\mathrm{d}t=R_A,\\ PNU_A=d\end{cases} \tag{8}$$

where $\lambda(t)=\begin{cases}\lambda(t),\text{at each time point } t, \text{ where } \lambda(t)\ge 0,\\ 0,\text{ if the reverse is true .}\end{cases}$

The physical meaning of the conjugate variables $\lambda(t)$ is expressed in terms of normal force exerted by constraints upon a structure. According to relation (2), the time law $\lambda(t)$ is approximated by a sum of T-periodic functions in the time interval $[0,T]$:

$$\lambda\cong\hat{N}\Lambda \tag{9}$$

Replacing the constraint $PNU_A=d$ at each time point of the interval $[0,T]$ by the projections of this constraint upon the subspace of the same functions, the superposition of which leads to the approximation of the time law λ, we obtain

$$\int_0^T N^{\mathrm{T}}PN\,\mathrm{d}t\,U_A=\int_0^T\hat{N}^{\mathrm{T}}d\,\mathrm{d}t. \tag{10}$$

Taking account of (10), set (8) is presented as

$$\begin{cases} \boldsymbol{A}\boldsymbol{U}_A+\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\Lambda}=\boldsymbol{R}_A, \\ \boldsymbol{B}\boldsymbol{U}_A=\boldsymbol{D}, \end{cases} \tag{11}$$

where

$$\boldsymbol{B}=\int_0^{\mathrm{T}}\hat{\boldsymbol{N}}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{N}\,\mathrm{d}t\,,\quad \boldsymbol{D}=\int_0^{\mathrm{T}}\hat{\boldsymbol{N}}^{\mathrm{T}}\boldsymbol{d}\,\mathrm{d}t\,.$$

From the system (11), $\boldsymbol{\Lambda}$ can be expressed as follows:

$$\boldsymbol{\Lambda}=\left(\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{R}_A-\boldsymbol{D}\right). \tag{12}$$

If the time law $\lambda(t)$ is approximated by relation (9), it may occur that at some time points $\lambda<0$, and it can cause negative normal interaction forces. In order to exclude negative values of λ, the corresponding constraints should be regarded as being inactive. Therefore, we consider the rows of the constraint matrix as the time function $\boldsymbol{P}=\boldsymbol{P}(t)$ and define them as zero values at the time points when the corresponding negative elements of the vector $\boldsymbol{\lambda}$ are obtained from relation (9). In general, iteration is necessary for determining the active and inactive constraints.

When the values of $\boldsymbol{\lambda}(t)$ are known, the generalized amplitude vector is determined by the relation

$$\boldsymbol{U}_A=\boldsymbol{A}^{-1}\left(\boldsymbol{R}_A-\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\Lambda}\right). \tag{13}$$

Analysis in the frequency domain employing the harmonic balance techniques may be considered as a special case of the weighted residual approach. The harmonic functions are employed as weighting functions $\boldsymbol{N}(t),\hat{\boldsymbol{N}}(t)$

$$\begin{aligned} \boldsymbol{N}(t)&=\left[\boldsymbol{I}\;\; \boldsymbol{I}\cos\omega t\;\; \boldsymbol{I}\sin\omega t\;\; \boldsymbol{I}\cos 2\omega t\;\; \boldsymbol{I}\sin 2\omega t\ldots\right], \\ \hat{\boldsymbol{N}}(t)&=\left[\hat{\boldsymbol{I}}\;\; \hat{\boldsymbol{I}}\cos\omega t\;\; \hat{\boldsymbol{I}}\sin\omega t\;\; \hat{\boldsymbol{I}}\cos 2\omega t\;\; \hat{\boldsymbol{I}}\sin 2\omega t\ldots\right], \end{aligned} \tag{14}$$

where $\boldsymbol{I}$ - the unity matrix of the dimension equal to the length of the vector $\boldsymbol{U}$, $\hat{\boldsymbol{I}}$ - the unity matrix of the dimension equal to the number of constraints, i.e., to the number of rows of the matrix $\boldsymbol{P}$, and $T=2\pi/\omega$. The generalized amplitude vector consists of the sine and cosine Fourier amplitudes

$$\boldsymbol{U}_A=\left(\boldsymbol{U}^0\,\boldsymbol{U}_c^1\,\boldsymbol{U}_s^1\boldsymbol{U}_c^2\,\boldsymbol{U}_s^2\ldots\right)^{\mathrm{T}}.$$

The finite element approach in time domain is obtained by dividing the interval $[0,T]$ into m finite elements, each of the length T/m. When employing the second order elements with three nodes, the generalized amplitude vector on the i-th element is presented as

$$\boldsymbol{U}=\boldsymbol{N}\boldsymbol{U}_A^i=\left[\boldsymbol{I}N_0,\;\boldsymbol{I}N_1,\boldsymbol{I}N_2\right]\begin{pmatrix}\boldsymbol{U}_{2i-1}\\ \boldsymbol{U}_{2i}\\ \boldsymbol{U}_{2i+1}\end{pmatrix}, \tag{15}$$

where $N_0=\dfrac{\xi(\xi-1)}{2},\; N_1=-(\xi-1)(\xi+1),\; N_2=\dfrac{\xi(\xi+1)}{2},\; \xi=\dfrac{t}{T}$.

During each iteration the matrices for the whole time interval $[0,T]$ are obtained according to the general rules for assembling finite element matrices, taking into account the periodic motion condition as $\boldsymbol{U}_1=\boldsymbol{U}_{2m+1}$. The global generalized amplitude vector is

$$\boldsymbol{U}_A=\left(\boldsymbol{U}_1,\boldsymbol{U}_2,\boldsymbol{U}_3,\ldots,\boldsymbol{U}_{2m}\right)^{\mathrm{T}}.$$

Transient analysis of nonlinear vibration in terms of slow varying amplitudes

Consider the matrix equation of motion of an elastic structure with nonlinear interaction (1) with the periodic excitation $\boldsymbol{R}(t)=\boldsymbol{R}(t+T)$. We present the displacements and forces as truncated Fourier series as

$$\begin{aligned} \boldsymbol{U}(t)&\cong\sum_{k=1}^{p}\boldsymbol{U}_c^k\cos(k-1)\omega t+\boldsymbol{U}_s^k\sin(k-1)\omega t\,, \\ \boldsymbol{R}(t)&\cong\sum_{k=1}^{p}\boldsymbol{R}_c^k\cos(k-1)\omega t+\boldsymbol{R}_s^k\sin(k-1)\omega t\,, \\ \boldsymbol{W}(t)&\cong\sum_{k=1}^{p}\boldsymbol{W}_c^k\cos(k-1)\omega t+\boldsymbol{W}_s^k\sin(k-1)\omega t\,, \end{aligned} \tag{16}$$

where according to the definition $\boldsymbol{U}_s^1=\boldsymbol{R}_s^1=\boldsymbol{W}_s^1=0$,, and $\omega=2\pi/T$. Employing the time averaging techniques, we consider the amplitudes $\boldsymbol{U}_c^k,\,\boldsymbol{U}_s^k,\,k=\overline{1,p}$ as time functions. After some transformation we obtain the equation of motion in terms of amplitudes slowly varying in time:

$$\begin{bmatrix} 0 & & & & & \\ & 2\omega\boldsymbol{M} & 0 & & & \\ & 0 & 2\omega\boldsymbol{M} & & & \\ & & & \ddots & & \\ & & & & 2(p-1)\omega\boldsymbol{M} & 0 \\ & & & & 0 & 2(p-1)\omega\boldsymbol{M} \end{bmatrix}\times\begin{pmatrix}\dot{\boldsymbol{U}}_c^0\\ \boldsymbol{U}_c^1\\ \boldsymbol{U}_s\\ \vdots\\ \dot{\boldsymbol{U}}_c^p\\ \boldsymbol{U}_s^p\end{pmatrix}=$$

$$
=\begin{bmatrix} -K & & & & & \\ & -\omega C & K-\omega^2 M & & & \\ & -K+\omega^2 M & -\omega C & & & \\ & & & \ddots & & \\ & & & & -(p-1)\omega C & K-\omega^2(p-1)^2 M \\ & & & & -K+\omega^2(p-1)^2 M & -(p-1)\omega C \end{bmatrix}\times
$$

$$
\times\begin{pmatrix} U_c^0 \\ U_c^0 \\ U_s^1 \\ \vdots \\ U_c^p \\ U_s^p \end{pmatrix}+\begin{pmatrix} W_c^0 \\ -W_s^1 \\ W_c^1 \\ \vdots \\ -W_s^p \\ W_c^p \end{pmatrix}=\begin{pmatrix} R_c^0 \\ -R_s^1 \\ R_c^1 \\ \vdots \\ -R_s^p \\ R_c^p \end{pmatrix}. \tag{17}
$$

The matrices in equation (17) are block-diagonal. With no nonlinearities present ($W \equiv 0$), equation (17) decomposes into k independent equations, each of which enables to obtain the corresponding Fourier amplitudes. If nonlinearities are present, it is necessary to solve equation (17) of the dimension $2(p-1)n \times 2(p-1)$.

If nonlinear interaction forces are concentrated in the localized points or zones of the structure, the vectors and matrices can be presented in a block form, where blocks correspond to the linear and nonlinear parts of the structure. The matrix inverse and other block operations corresponding to the linear part of the structure are carried out during an initial operation. As a result, during each iteration the nonlinear matrix equation of the dimension $2(p-1)n_2 \times 2(p-1)n_2$ is to be considered, where n -is the number of the nonlinear degrees of freedom.

If the simple iteration scheme for solving the nonlinear equation at each numerical integration step doesn't converge, the Newton-Raphson iteration is necessary. In this case at eachiteration the derivative matrix $\partial W_A / \partial U_A$ is to be determined.

In the case of ill-conditioned stiffness matrix, i.e., if rigid body motions of the structure are possible, set (17) cannot be solved, because it is necessary to invert the matrix K at the very first iteration. As a way out from this situation, the equivalent transformation of the matrix K and of the nonlinear term W can be accomplished by presenting the left upper block of equation

(17) as

$$
0=-\hat{K}U_c^0+\hat{W}_c^0+R_c^0, \tag{17'}
$$

where $\hat{K}=K+\max_{i,j}|k_{ij}|I$, $W_c^0=\hat{W}_c^0+\max_{i,j}|k_{ij}|U_c^0$, with the nonsingular matrix $\hat{K}$.

Stability analysis of the vibration laws

The steady motion laws are always stable when obtained by numerically integrating the equations of motion until transient vibrations cease. However, some unstable laws can appear among the motion laws obtained by employing the weighted residual approach. The stability of the motion law is to be checked by linearizing the nonlinear equation at the solution point and by applying well known transfer matrices or Hill's determinant techniques, or it is possible to investigate the stability of the motion laws by employing time-averaged equations. In the latter case the linearized equation for checking if the solution U_A^* is stable is obtained as

$$
\dot{X}_A=B^{-1}\left(A+\left.\frac{\partial U_A}{\partial W_A}\right|_{U_A^*}\right)X_A. \tag{18}
$$

where the left-hand side matrix of equation (17) is denoted by B. To investigate stability it is sufficient to evaluate the signs of the roots of the characteristic equation of obtained from the differential equation (18), employing the Routh-Hurwitz criterion.

Externally excited impact vibration of a rod-typevibroconverter loaded by a longitudinal force

A rod-type vibroconverter (VC) presented in Fig.1 isemployed in vibrodrives for creating a varying normal interaction force. A VC of the length l is presented by a finite element model, the loading force being P_0 and a longitudinal harmonic excitation force is $P_1(t)=P_1 \sin\omega t$. The computed

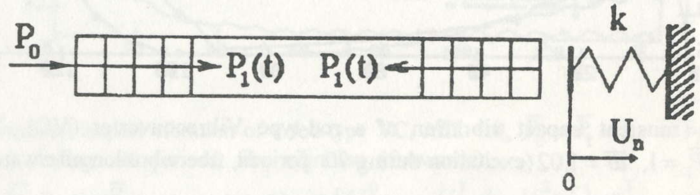


Fig.1. Finite element model of a vibroconverter.

results are presented by employing the dimensionless quantities

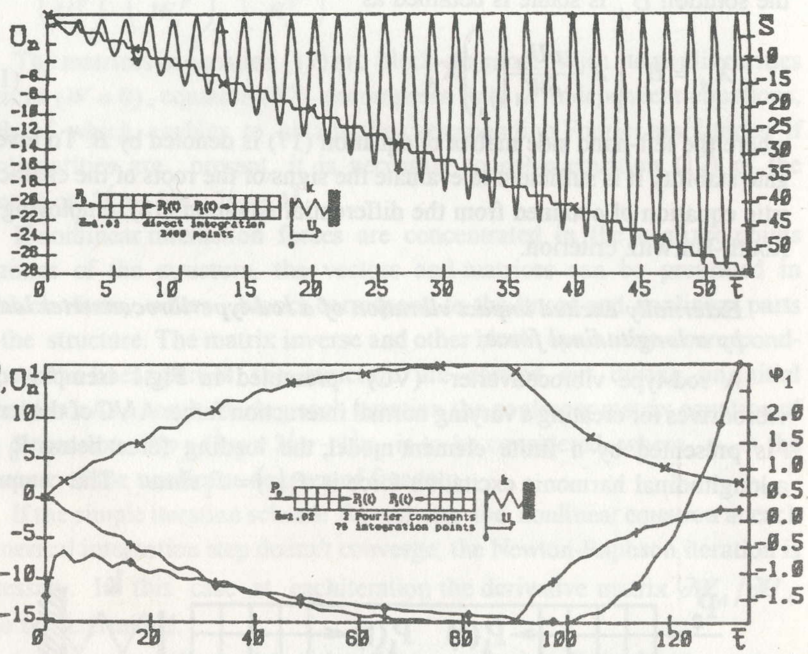
$$\bar{t}=\frac{t}{l}\left(\frac{E}{\rho}\right)^{\frac{1}{2}},\ \bar{U}=\frac{U}{l},\ \dot{\bar{U}}=\dot{U}\left(\frac{\rho}{E}\right)^{\frac{1}{2}},\ \bar{k}=\frac{k}{El},\ \bar{P}=\frac{P}{EF},\ \omega=\omega l\left(\frac{\rho}{E}\right)^{\frac{1}{2}}$$

where E, ρ are the Young's modulus and the density of the material. The dimensionless impetus of normal contact interaction forces is obtained from the relation

$$\bar{S}=\frac{S}{EFl}\left(\frac{\rho}{E}\right)^{\frac{1}{2}}.$$

A mechanical Q-factor of the VC is assumed to be equal to fifty, i.e., Q=50.

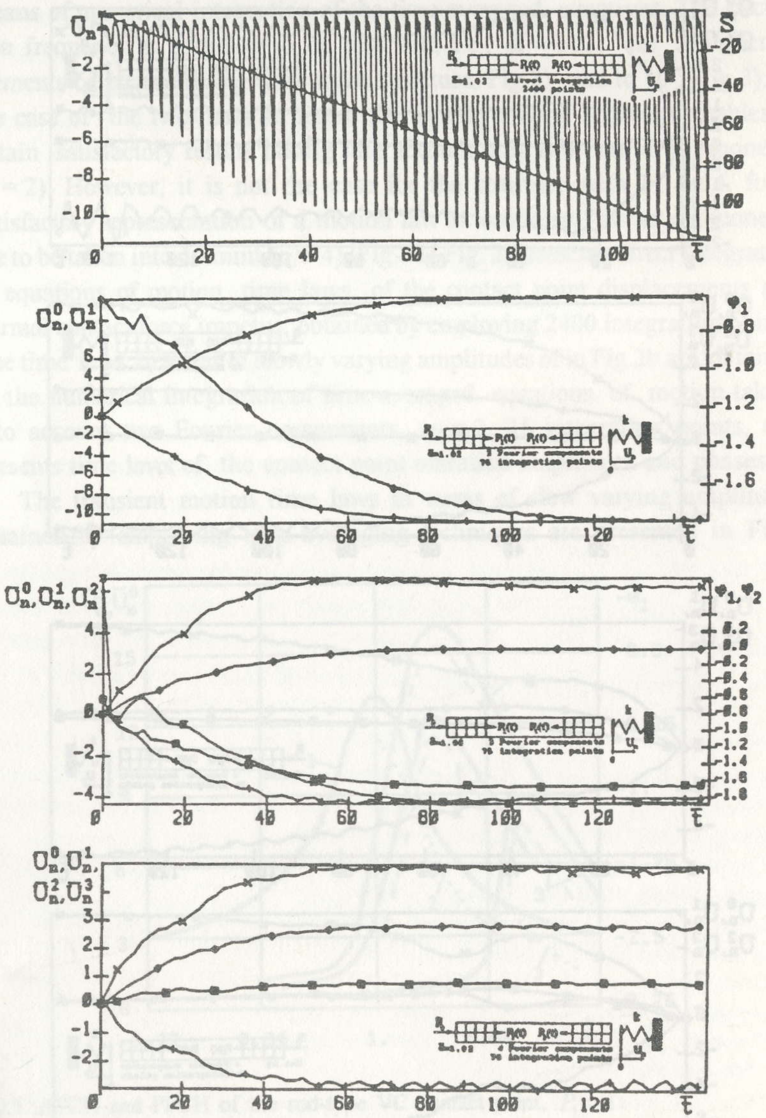
Fig.2 and Fig.3 present the time laws of the right-hand end displacements of the VC obtained by means of direct numerical integration of the equations



14

Fig.2 Transient impact vibration of a rod-type Vibroconverter (VC), NEL=1, $\bar{P}_0=\bar{P}_1=1$, $\bar{\omega}=1.02$ (excitation during 90 periods, free vibration afterwards):

$$+-U_c^0,\ \times-U^1=\left(\left(U_c^1\right)^2+\left(U_s^1\right)^2\right)^{\frac{1}{2}},\ \lozenge-\varphi^1=\mathrm{arctg}\left(U_c^1/U_s^1\right).$$

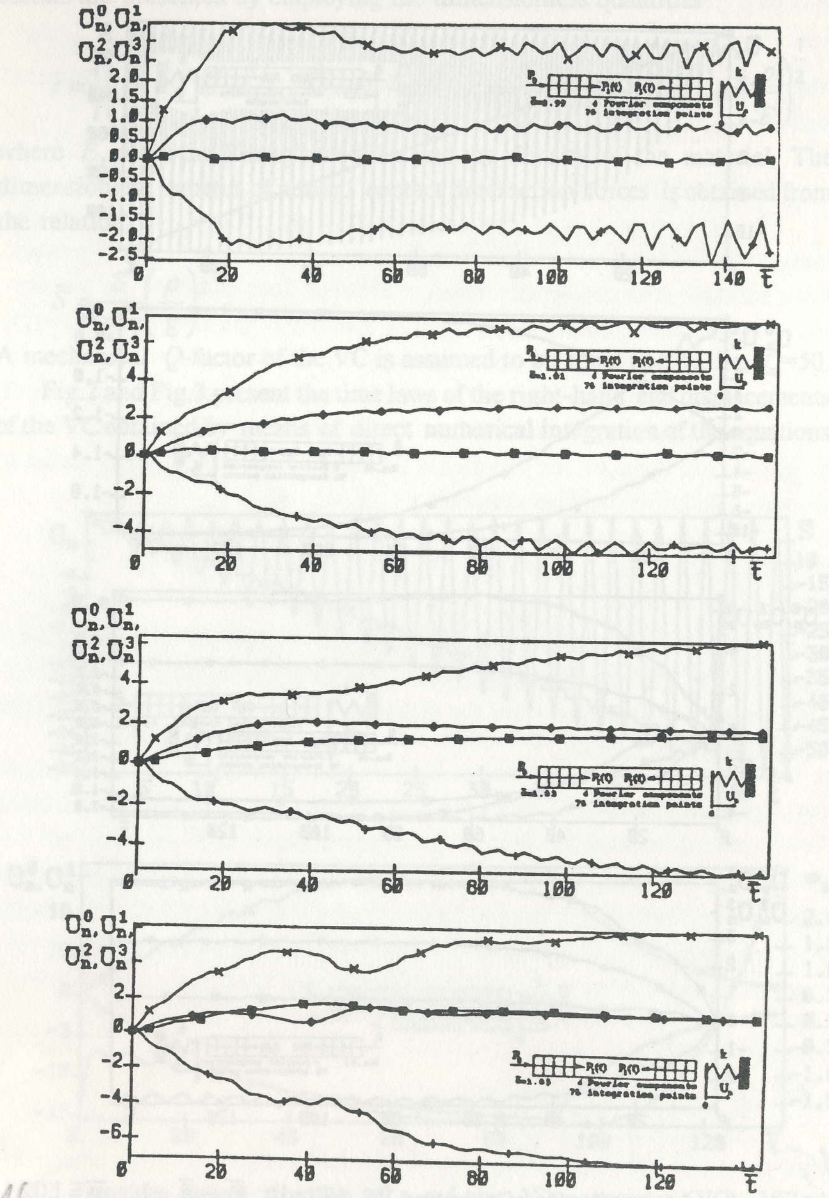


15

Fig.3. Transient impact vibration of a rod-type VC, NEL=10, $\bar{P}_0=\bar{P}_1=1$, $\bar{\omega}=1.02$ (excitation during 90 periods, free vibration afterwards):

a) $+-\bar{U}_n$; $\times-\bar{S}$; c) $+-U_c^0$; $\times-U^1$; $\lozenge-U^2$; $\square-\varphi^1$; $-\varphi^2$;

b) $+-U_c^0$; $\times-U^1$; $\lozenge-\varphi^1$; d) $+-U_c^0$; $\times-U^1$; $\lozenge-U^2$; $\square-U^3$.

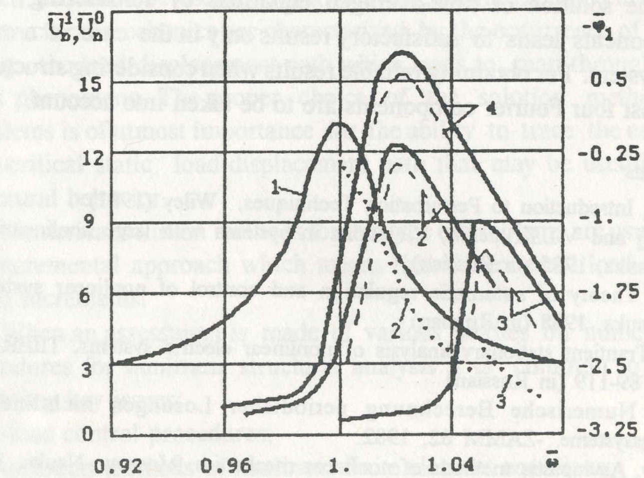


16

Fig.4. Transient impact vibration of a rod-type VC at several values of the excitation frequency, NEL=10, $\overline{P}_0 = \overline{P}_1 = 1$, $(+-U_c^0;\ \times-U^1;\ \diamond-U^2;\ \square-U^3)$
a) $\overline{\omega} = 0.99$; b) $\overline{\omega} = 1.01$; c) $\overline{\omega} = 1.03$; d) $\overline{\omega} = 1.05$

of motion, and the time laws of Fourier component amplitudes obtained by means of numerical integration of the time-averaged equations. The excitation frequency is assumed to be resonant, i.e., $\overline{\omega} = 1.02$, the number of elements being equal to 1 (two-mass structure, Fig.2) and to 10 (Fig.3). In the case of the two-mass structure the time-averaging approach enables to obtain satisfactory results taking into account only two Fourier components ($p = 2$). However, it is not the case for the structure with 11 d.o.f. for a satisfactory representation of a motion law at least four Fourier components are to be taken into account ($p = 4$), Fig.3d. Fig. 2a presents direct integration of equations of motion, time laws of the contact point displacements and normal contact force impetus, obtained by employing 2400 integration points. The time laws in terms of slowly varying amplitudes of in Fig.2b are obtained by the numerical integration of time-averaged equations of motion taking into account two Fourier components, $p = 2$, 75 integration points, and presents time laws of the contact point vibration amplitudes and phases.

The transient motion time laws in terms of slow varying amplitudes obtained by employing time-averaging techniques are presented in Fig.4



17

Fig.5. AFCH and PFCH of the rod-type VC contact point, $\overline{P}_1 = 1$:
1 -unconstrained vibration;
2 -$\overline{P}_0 = 1$, $p = 2$, $\overline{k} = 10$, $NEL = 10$;
3 -$\overline{P}_0 = 1.5$, $p = 2$, $\overline{k} = 10$, $NEL = 10$;
4 -$\overline{P}_0 = 1$, $p = 2$, $\overline{k} = 10$, $NEL = 1$;
—— amplitude U^1; - - - - amplitude U^0, ... phase φ^1

taking into account four Fourier components. In Fig. 5 the AFCH and PFCH are presented by several values of the load P. As a distinctive feature a resonant frequency shift to the right in comparison with the resonant frequency of an unconstrained VC and different steepness of slopes of the AFCH is observed. Curves 1 and 4 show the difference obtained in the case of one-element and ten element structures.

Conclusions

1. For the analysis of the steady motion law of structural equations with nonlinear terms with unilateral constraints uponthe displacements the weighted residual approach has been employed, and the well-known harmonic balance method can be regarded as a special case. Motion laws in terms of the slow varying amplitudes have been obtained by means of the time averaging techniques. When integrating the averaged equations numerically, harmonic components are obtained at each time station by means of Fourier transformation.

2. The study of longitudinal impact vibrations of a rod-type vibroconverter shows that the solution of time-averaged equations by considering two Fourier components leads to satisfactory results only in the case of a two-mass elastic system. For obtaining reliable results when considering structural models, at least four Fourier components are to be taken into account.

References

1. A.H.Nayfeh, Introduction to Perturbation Techniques, Wiley (1981).
2. V.I. Babickij and V.L.Krupenin, Vibrations in systems with large nonlinearities, Moscow, Nauka, 1985 (in Russian)
3. E.P.Popov, Theory of automatic regulation and control of nonlinear systems, Moscow, Nauka, 1988 (in Russian)
4. S.Skelboe, Trantient stationary analysis of nonlinear electric systems, TIIER, 70 (10), 1982, 89-119 (in Russian)
5. F.H.Ling, Numerische Berechnung periodisher Losungen nichtlinearer Schwingungssysteme. -ZAMM 62, 1982.
6. N.I.Moisejev, Asymptotic methods of nonlinear mechanics, Moscow, Nauka, 1969 (in Russian)
7. R.Barauskas, Analysis of regimes of multiperiodic and stationary vibrations in mechanical systems with essential nonlinearities, Vibrotechnics, 67, 1991 (in Russian)

Received 6 May 1992